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# On the graded quotients of the ring of Fricke characters of a free group

(畠中英里*氏（東京農工大学）との共同研究)

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**Abstract**

In this paper, for a group $G$, we consider an $\mathrm{Aut}\, G$-invariant ideal $J$ generated by $\mathrm{tr}\, x-2$ for any $x \in G$ in the ring of Fricke characters of $G$. We study a descending filtration $J \supset J^2 \supset J^3 \supset \cdots$, and its graded quotients $\mathrm{gr}^k(J) := J^k/J^{k+1}$ for $k \geq 1$. The first purpose of this paper is to determine the structure of $\mathrm{gr}^k(J)$ if $G$ is a free group $F_n$ of rank $n$ and $k=1,2$.

Next, we introduce a normal subgroup $\mathcal{E}_G(k)$ consisting of automorphisms of $G$ which act on $J/J^{k+1}$ trivially. These normal subgroups define a central filtration of $\mathrm{Aut}\, G$. This is a Fricke character analogue of the Andreadakis-Johnson filtration $\mathcal{A}_G(k)$ of $\mathrm{Aut}\, G$. The main purpose of the paper is to show that $\mathcal{E}_{F_n}(1)$ is equal to $\mathrm{Inn}\, F_n \cdot \mathcal{A}_{F_n}(2)$ where $\mathrm{Inn}\, F_n$ is the inner automorphism group of a free group $F_n$, and that $\mathcal{A}_{F_n}(2k) \subset \mathcal{E}_{F_n}(k)$ for any $k \geq 1$.

Let $G$ be a group generated by elements $x_1, \ldots, x_n$. We denote by

$$R(G) := \mathrm{Hom}(G, \mathrm{SL}(2, \mathbf{C}))$$

the set of all group homomorphisms from $G$ to $\mathrm{SL}(2, \mathbf{C})$. Let

$$\mathcal{F}(R(G), \mathbf{C}) := \{\chi : R(G) \to \mathbf{C}\}$$

be the set of all complex-valued functions of $R(G)$. Then we can consider $\mathcal{F}(R(G), \mathbf{C})$ as a commutative ring in a natural way. For any $x \in G$, we define an element $\mathrm{tr}\, x \in \mathcal{F}(R(G), \mathbf{C})$ to be

$$(\mathrm{tr}\, x)(\rho) := \mathrm{tr}\, \rho(x)$$

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for any $\rho \in R(G)$. Here "tr" in the right hand side means the trace of $2 \times 2$ matrix $\rho(x) \in \mathrm{SL}(2, \mathbf{C})$. The element $\operatorname{tr} x$ in $\mathcal{F}(R(G), \mathbf{C})$ is called the Fricke character of $x \in G$. Let $\mathfrak{X}(G)$ be the $\mathbf{Z}$-submodule of $\mathcal{F}(R(G), \mathbf{C})$ generated by all $\operatorname{tr} x$ for $x \in G$. Then $\mathfrak{X}(G)$ is closed under the multiplication of $\mathcal{F}(R(G), \mathbf{C})$.

Classically, Fricke characters were begun to studied by Fricke for a free group $F_n$ on $x_1, \ldots, x_n$ in connection with certain problems in the theory of Riemann surfaces. (See [3].) In 1970, Horowitz [5] and [6] investigated algebraic properties of $\mathfrak{X}(G)$ using the combinatorial group theory. In particular, he [5] showed that for any $x \in G$, the Fricke character $\operatorname{tr} x$ can be written as a polynomial with integral coefficients in $2^n - 1$ characters $\operatorname{tr} x_{i_1} x_{i_2} \cdots x_{i_l}$ for $1 \leq l \leq n$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. He [6] also showed that the subgroup of $\operatorname{Aut} F_n$ consisting of automorphisms which act on $\mathfrak{X}(F_n)$ tirivially is just the inner automorphism group $\operatorname{Inn} F_n$ of $F_n$. Namely, the action of $\operatorname{Aut} F_n$ on the ring of Fricke characters $\mathfrak{X}(F_n)$ induces a faithful representatrion of the outer automorphism group $\operatorname{Out} F_n := \operatorname{Aut} F_n / \operatorname{Inn} F_n$. However, since the rank of $\mathfrak{X}(F_n)$ as a $\mathbf{Z}$-module is not finite in general, it is not so easy to study this representation directly.

On the other hand, in order to make the structure of the Fricke characters $\mathfrak{X}(F_n)$ clear, it is important to study the ideal of polynomials in the characters which vanish on any representations of $G$. More precisely, consider a polynomial ring

$$\mathbf{Z}[t] := \mathbf{Z}[t_{i_1 \cdots i_l} \mid 1 \leq l \leq n, \ 1 \leq i_1 < i_2 < \cdots < i_l \leq n]$$

of $2^n - 1$ indeterminates, and an ideal

$$I = \{ f \in \mathbf{Z}[t] \mid f(\operatorname{tr} \rho(x_{i_1} \cdots x_{i_l})) = 0 \ \text{ for any } \ \rho \in R(G) \}.$$

In [5], for $G = F_n$, Horowitz showed that $I$ is trivial for $n = 1$ and 2, and is principal for $n = 3$. Whittemore [17] showed that $I$ is not principal for $G = F_n$ and $n \geq 4$. Although the ideal $I$ has been studied by many authors for over forty years, very little is known for it.

Here, we consider the rationalization of the situation above. Let $\mathfrak{X}_{\mathbf{Q}}(G)$ be a $\mathbf{Q}$-subspace of $\mathcal{F}(R(G), \mathbf{C})$ generated by $\operatorname{tr} x$ for any $x \in G$. Similary to $\mathfrak{X}(G)$, $\mathfrak{X}_{\mathbf{Q}}(G)$ is closed under the multiplication of $\mathcal{F}(R(G), \mathbf{C})$, and has a multiplicative unit $1 = \frac{1}{2}\operatorname{tr} 1_G$. Hence, $\mathfrak{X}_{\mathbf{Q}}(G)$ is a ring. We call $\mathfrak{X}_{\mathbf{Q}}(G)$ the ring of Fricke characters of $G$ over $\mathbf{Q}$. By a result of Horowitz, we see that for any $x \in G$, the Fricke character $\operatorname{tr} x$ can be written as a polynomial with ratinal coefficients in $n + \binom{n}{2} + \binom{n}{3}$ characters $\operatorname{tr} x_{i_1} x_{i_2} \cdots x_{i_l}$ for $1 \leq l \leq 3$ and $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. Consider a polynomial ring

$$\mathbf{Q}[t] := \mathbf{Q}[t_{i_1 \cdots i_l} \,|\, 1 \leq l \leq 3,\ 1 \leq i_1 < i_2 < \cdots < i_l \leq n]$$

and its ideal

$$I_{\mathbf{Q}} := \{ f \in \mathbf{Q}[t] \,|\, f(\operatorname{tr} \rho(x_{i_1} \cdots x_{i_l})) = 0 \ \text{ for any } \ \rho \in R(G) \}.$$

Similarly to $I$, the ideal $I_{\mathbf{Q}}$ plays important roles in the various study of the ring structure of $\mathfrak{X}_{\mathbf{Q}}(G)$. One of the most advantages to consider the rationalization of the Fricke characters is that the number of the indeterminates of $\mathbf{Q}[t]$ is fewer than that of $\mathbf{Z}[t]$, and it makes various computation much easy to handle.

In the present paper, in order to construct finite dimensinal representations of $\operatorname{Aut} G$, we consider a descending filtration of $\operatorname{Aut} G$-invariant ideals of $\mathbf{Q}[t]/I_{\mathbf{Q}}$, and take its graded quotients. Set $t'_{i_1 \cdots i_l} := t_{i_1 \cdots i_l} - 2 \in \mathbf{Q}[t]$. We also denote by $t'_{i_1 \cdots i_l}$ its coset class in $\mathbf{Q}[t]/I_{\mathbf{Q}}$. Consider an ideal

$$J := (t'_{i_1 \cdots i_l} \,|\, 1 \leq l \leq 3,\ 1 \leq i_1 < i_2 < \cdots < i_l \leq n) \subset \mathbf{Q}[t]/I_{\mathbf{Q}}$$

generated by all $t'_{i_1 \cdots i_l}$'s. Then, we have a descending filtration

$$J \supset J^2 \supset J^3 \supset \cdots$$

of $\operatorname{Aut} G$-invariant ideals of $\mathbf{Q}[t]/I_{\mathbf{Q}}$. Set

$$\operatorname{gr}^k(J) := J^k / J^{k+1}.$$

Each of $\operatorname{gr}^k(J)$ is $\operatorname{Aut} G$-invariant $\mathbf{Q}$-vector space of finite dimension for any $k \geq 1$. This technique is deeply inspired by a result of Magnus [12] who originally studied the behavior of the action of $\operatorname{Aut} F_3$ on $\operatorname{gr}^1(J)$. In [12], he pointed out the difficulties to find $\operatorname{Aut} F_n$-invariant ideals of $\mathfrak{X}(F_n)$ and its quotient rings as a finite dimensional representation of $\operatorname{Aut} F_n$ in general. Moreover, he [12] also stated that in order to get accessible situation, it seems to be better to use rational functions rather than integral polynomials. In this paper, however, we consider the rational polynomials to obtain finite dimensional representations of Aut $F_n$.

The first purpose of the paper is to determine the structure of $\mathrm{gr}^k(J)$ for $G = F_n$, $n \geq 3$ and $k = 1, 2$. Set

$$T := \{t'_i \,|\, 1 \leq i \leq n\} \cup \{t'_{ij} \,|\, 1 \leq i < j \leq n\} \cup \{t'_{ijk} \,|\, 1 \leq i < j < k \leq n\} \subset J$$

and

$$\begin{aligned} S :=& \{t'_i t'_j \,|\, 1 \leq i \leq j \leq n\} \cup \{t'_i t'_{ab} \,|\, 1 \leq i \leq n,\ 1 \leq a < b \leq n\} \\ & \cup \{t'_i t'_{abc} \,|\, 1 \leq i \leq n,\ 1 \leq a < b < c \leq n\} \\ & \cup \{t'_{ij} t'_{ab} \,|\, 1 \leq i < j \leq n,\ 1 \leq a < b \leq n,\ (i,j) \leq (a,b)\}, \\ & \cup \{t'_{ab} t'_{abc},\ t'_{ac} t'_{abc},\ t'_{bc} t'_{abc} \,|\, 1 \leq a < b < c \leq n\} \\ & \cup \{t'_{ia} t'_{abc}, t'_{ib} t'_{abc}, t'_{ic} t'_{abc}, t'_{ia} t'_{ibc}, t'_{ab} t'_{iac}, t'_{ab} t'_{ibc}, t'_{ac} t'_{ibc}, t'_{ib} t'_{iac} \\ & \qquad \,|\, 1 \leq i < a < b < c \leq n\} \\ & \cup \{t'_{ja} t'_{ibc}, t'_{jb} t'_{iac}, t'_{jc} t'_{iab}, t'_{ab} t'_{ijc}, t'_{ac} t'_{ijb}, t'_{bc} t'_{ija} \\ & \qquad \,|\, 1 \leq i < j < a < b < c \leq n\} \\ & \subset J^2 \end{aligned}$$

respectively. We show

**Theorem 1.** *For $G = F_n$ and $n \geq 3$, the sets $T$ and $S$ are basis of the $\mathbf{Q}$-vector spaces $\mathrm{gr}^1(J)$ and $\mathrm{gr}^2(J)$ respectively.*

In general, it seems to be very complicated to find a basis of $\mathrm{gr}^k(J)$ for general $k \geq 3$.

Next, for any group $G$, we consider a descending filtration of Aut $G$. For any $k \geq 1$, let $\mathcal{E}_G(k)$ be the subgroup of Aut $G$ consisting of automorphisms which act on $J/J^{k+1}$ trivially. Then we see that the groups $\mathcal{E}_G(k)$ define a descending filtration

$$\mathcal{E}_G(1) \supset \mathcal{E}_G(2) \supset \cdots \supset \mathcal{E}_G(k) \supset \cdots$$

of Aut $G$.

This filtration is a Fricke character analogue of the Andreadakis-Johnson filtration $\mathcal{A}_G(k)$ of Aut $G$. The Andreadakis-Johnson filtration was originally introduced by Andreadakis [2] in 1960's. In a series of his pioneer works [7], [8], [9] and [10], Johnson established the theory of Johnson homomorphisms in the study of the mapping class of surfaces. Togather with the theory of the Johnson homomorphisms, the Andreadskis-Johnson filtration is one of powerful tools to study the group structure of the automorphism group of a group. (See [14] or [15] for basic materials concerning the Andreadakis-Johnson filtration and the Johnson homomorphisms.)

The main purpose of the paper is to show

**Proposition 1.** *For any* $k, l \geq 1$, $[\mathcal{E}_G(k), \mathcal{E}_G(l)] \subset \mathcal{E}_G(k+l)$.

and

**Theorem 2.** *For any* $n \geq 3$,

1. $\mathcal{E}_{F_n}(1) = \mathrm{Inn}\, F_n \cdot \mathcal{A}_{F_n}(2)$.
2. $\mathcal{A}_{F_n}(2k) \subset \mathcal{E}_{F_n}(k)$.

From Proposition 1, we see that $\{\mathcal{E}_G(k)\}$ is a central filtration of $\mathcal{E}_G(1)$. Then a natural problem to consider is how different is $\{\mathcal{E}_G(k)\}$ from the Andreadakis-Johnson filtration $\{\mathcal{A}_G(k)\}$. The partial answer to this question for $G = F_n$ is the theorem above.

On the other hand, since $\{\mathcal{E}_G(k)\}$ is central, each of the graded quotient $\mathrm{gr}^k(\mathcal{E}_{F_n}) := \mathcal{E}_G(k)/\mathcal{E}_G(k+1)$ is an abelian group. At the end of the paper, we show

**Theorem 3.** *For any* $n \geq 3$,

1. *Each of* $\mathrm{gr}^k(\mathcal{E}_{F_n})$ *is torsion-free.*
2. $\dim_{\mathbf{Q}}(\mathrm{gr}^k(\mathcal{E}_{F_n}) \otimes_{\mathbf{Z}} \mathbf{Q}) < \infty$.

To show this, we introduce Johnson homomorphism like homomorphisms $\eta_k$. Observing Theorem 2, we see that $\mathrm{gr}^1(\mathcal{E}_{F_n})$ is finitely generated. In general, however, it seems to be quite a difficult to determine the structure of $\mathrm{gr}^k(\mathcal{E}_{F_n})$ even the case where $k = 1$.

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